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COMPLETENESS IN AN ORDERED FIELD

By

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B.S., University of New Mexico, 1963

Presented in partial fulfillment of the requirements
for the degree of

Master of Arts for Teachers of Mathematics

UNIVERSITY OF MONTANA

1968

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ACKNOWLEDGEMENTS

I wish to express my appreciation to Professor Gloria C. Hewitt for her interest, guidance, and instruction throughout the preparation of this thesis. Also, I wish to thank Professors William M. Myers and Kenneth V. Lottick for their critical reading of the thesis. This was written while I was a participant in the 1967-68 National Science Foundation Academic Year Institute for Mathematics Teachers.

R. H. M.



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TABLE OF CONTENTS





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## LIST OF NOTATION

N . . . . . . . . the set of natural numbers, {1, 2, 3, ...}

Z . . . . . . . . the set of all integers

Q . . . . . . . . the field of rational numbers

R . . . . . . . . the field of real numbers

Ø . . . . . . . . the empty set

A - B . . . . . . the set of elements in A but not in B



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# INTRODUCTION

In essence this paper is an analysis of the real number system. We assume that the reader is familiar with the construction of R, the field of real numbers, and its properties as a complete Archimedean ordered field. We also assume that the reader is aware that Q, the field of rational numbers, is an Archimedean ordered field which is not complete. It is sufficient for the context of this paper to know that the field of real numbers is complete in the sense that every non-empty subset which is bounded above has a least upper bound, and every Cauchy sequence converges. Precise definitions of these terms will be given. Working under these assumptions we investigate various notions of completeness from the general context of an ordered field. This enables us to analyze the properties individually, and to introduce systems which satisfy some but not all of them.

In the first chapter the basic definitions of an ordered field, Archimedean order, absolute value, sequences, and Cauchy Completeness, are given. Lemmas and theorems pertaining to these definitions lead to two principal results. The first is that the field of rational numbers may be considered a subfield of any ordered field. The second is that an Archimedean ordered field may be considered a subfield of the field of real

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numbers. Examples are included to serve as illustrations of the ideas under discussion. The chapter concludes with the description of a topology for an ordered field.

In the second chapter we introduce the definitions which pertain particularly to completeness. We show that certain completeness properties imply that the ordering is Archimedean. We find that not all Cauchy Complete ordered fields are Archimedean. We then consider the relationships between the various completeness properties showing that six are equivalent, and that in an Archimedean ordered field eight completeness properties are equivalent. We conclude by showing that R is essentially the only complete Archimedean ordered field. That is, we show that a complete Archimedean ordered field is order-isomorphic to the field of real numbers.

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## CHAPTER I

## ORDERED FIELDS AND ARCHIMEDEAN ORDER

The first portion of this chapter will be concerned with basic definitions and theorems. The following notation about elements of a field F will be observed. $\Theta$ will denote the additive identity and e, the multiplicative identity. If $x \in F$ then $-x$ will denote the additive inverse of x. If $x \neq \Theta$, then $x^{-1}$ will denote the multiplicative inverse of x. For all $x \in F$ we define $1 \cdot x = x$ and $0 \cdot x = \Theta$. For all $m \in N$, mx is defined inductively by $mx = (m - 1)x + x$. If $-m \in Z$ and $-m < 0$ then $-mx$ is defined by $-mx = m(-x)$. Appropriate adjustments will be made in this notation after showing that the field Q is a subfield of any ordered field.

The first definition describes the structure of an ordered field.

Definition 1.1: A non-empty subset P of a field F is called a positive class if it satisfies the following three properties:

(i) If $a \in P$ and $b \in P$, then $a + b \in P$.

(ii) If $a \in P$ and $b \in P$, then $ab \in P$.

(iii) If $a \in F$, then exactly one of the following holds: $a \in P$, $a = \Theta$, or $-a \in P$.

If a field F contains a positive class P, we say that F is ordered by P and that F is an ordered field.

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Before studying the consequences of this definition, it will be convenient to have an alternate definition.

Theorem 1.2: F is an ordered field if and only if a relation $<$ can be defined on F satisfying:

(i) If $x, y, z \in F$, and $x < y$, then $x + z < y + z$.

(ii) If $x, y, z \in F$, $\Theta < z$, and $x < y$, then $xz < yz$.

(iii) For all $x, y \in F$, exactly one of $x = y$, $x < y$, or $y < x$ holds.

(iv) If $x, y, z \in F$, $x < y$, and $y < z$, then $x < z$.

Proof: Part I: Suppose F is an ordered field, then F contains a positive class P. Define a relation $<$ on F by: $a < b$ if and only if $(b - a) \in P$.

(i) Let $x, y, z \in F$, then $x < y$ if and only if $(y - x) \in P$. But $(y + z) - (x + z) = y + z - x - z = y - x$. Thus $(y - x) \in P$ implies $(y + z) - (x + z) \in P$. That is, if $x < y$, then $x + z < y + z$.

(ii) Suppose $x, y, z \in F$, with $\Theta < z$, and $x < y$, then $(z - \Theta) = z \in P$ and $(y - x) \in P$; so $(y - x)z = (yz - xz) \in P$. Hence, $xz < yz$.

(iii) Suppose $x, y \in F$. Either $(y - x) \in P$, and hence $x < y$; or $(y - x) = \Theta$, and hence $x = y$; or $-(y - x) \in P$, and hence $y < x$.

(iv) Suppose $x, y, z \in F$, with $x < y$, and $y < z$. Then $y - x \in P$ and $z - y \in P$; hence $(z - y) + (y - x) \in P$. That is, $z - x \in P$; so $x < z$.

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Part II: Suppose there is a relation $<$ defined on the field F satisfying (i), (ii), (iii), and (iv). Let $P = \{x \mid x \in F \text{ and } \Theta < x\}$. If $a, b \in P$, then $\Theta < a$, and $\Theta < b$.

(i) $\Theta < a$ implies $\Theta + b < a + b$ which in turn implies $b < a + b$. So $\Theta < a + b$ and $a + b \in P$.

(ii) $\Theta \cdot b < a \cdot b$, so $\Theta < ab$. Thus $ab \in P$.

(iii) Let $a \in F$. Since $\Theta \in F$, either $a = \Theta$; or $\Theta < a$, hence $a \in P$; or $a < \Theta$ which implies $a - a < \Theta - a$ which in turn implies $\Theta < -a$, hence $-a \in P$.

Therefore P forms a positive class for F, and F is an ordered field.

Thus, given an ordered field F, one may assume that there exists a relation $<$ defined on F satisfying the conditions listed in theorem 1.2. As a matter of notation if $a, b \in F$, then $b < a$ will sometimes be written $a > b$. $a < b$ will be read "a is less than b", and $a > b$ will be read "a is greater than b". $a \leq b$ will be used when it is true that either $a < b$ or $a = b$; and $a \geq b$ will be used when it is true that $a > b$ or $a = b$. $a < b < c$ will be used to mean $a < b$ and $b < c$; and $a \leq b \leq c$ will be used to mean $a \leq b$ and $b \leq c$. When it is appropriate, the terms "maximum" and "minimum" will be used in the same sense that they are used in referring to the real numbers. Some basic properties of the relation $<$ are summarized in the next lemma.

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<u>Lemma 1.3</u>: If F is an ordered field with relation $<$, then:

(i) $\Theta < e$.

(ii) $\Theta < a$ implies $\Theta < a^{-1}$.

(iii) $m, n \in Z$ with $m < n$ in Z implies $me < ne$ in F.

(iv) $a < b$ and $c < d$ implies $a + c < b + d$.

(v) $a > b$ and $c < \Theta$ implies $ac < bc$.

(vi) $a < \Theta$ implies $a^{-1} < \Theta$.

Proof:

(i) If it is not true that $\Theta < e$, then since $e \neq \Theta$, $\Theta < -e$. $\Theta < -e$ implies $\Theta < (-e)(-e)$. $(-e)(-e) = e$; so $\Theta < -e$ implies $\Theta < e$. This is a contradiction, hence $\Theta < e$.

(ii) Assume $a^{-1} < \Theta$. Then $a^{-1}\cdot a < \Theta\cdot a$. That implies $e < \Theta$, contradicting (i), so $\Theta < a^{-1}$.

(iii) $m < n$ in Z implies $n - m > 0$ which implies $(n - m) \in N$. It will be sufficient to show that $n \in N$ implies $ne > \Theta$, as then we will have $(n - m)e > \Theta$ which will imply $ne - me > \Theta$, and in turn $ne > me$. $1\cdot e = e$, $e > \Theta$, and $ne = (n - 1)e + e$. Hence by induction, $ne > \Theta$.

(iv) $a < b$ implies $a + c < b + c$. $c < d$ implies $b + c < b + d$. Thus, $a + c < b + d$.

(v) $c < \Theta$ implies $\Theta < -c$ which with $b < a$ implies $b(-c) < a(-c)$. Consequently, $-bc < -ac$, which implies $ac < bc$.

(vi) Assume $a^{-1} > \Theta$, then $a^{-1}\cdot a < a^{-1}\cdot\Theta$ which means

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$e < \Theta$. This is a contradiction, thus $a^{-1} < \Theta$.

Before continuing, we discuss some examples. Both the field R of real numbers and the field Q of rational numbers are ordered fields ([1], p. 34).

There are fields which are not ordered fields. The field C of complex numbers is one such field. To see this, suppose P is a positive class for C. Since $i \neq 0$, either $i \in P$ or $-i \in P$. $i \in P$ implies $i^2 \in P$ which is the same as $-1 \in P$. This in turn implies $-i = -1 \cdot i \in P$ which is a contradiction. $-i \in P$ implies $(-i)^2 \in P$ which is the same as $-1 \in P$, leading to $i \in P$, which is another contradiction.

Another ordered field is the field Q(x). Elements of Q(x) are of the form f(x)/g(x) where f(x) and g(x) are polynomials with rational numbers as coefficients, and g(x) is not the zero polynomial. Addition and multiplication are defined in Q(x) as usual. The positive class P is the subset consisting of all elements of the form f(x)/g(x) in which the leading coefficient of the product $f(x) \cdot g(x)$ is a positive rational number.

A fourth example of an ordered field is developed in the next definition and subsequent theorems.

<u>Definition 1.4</u>: Let R be the field of real numbers and x an indeterminate. R<x> will denote the set of all expressions g where

$$g = \sum_{k=-\infty}^{\infty} a_k x^k, \text{ with } a_k \in R \text{ for all } k,$$

with the understanding that at most a finite number of the

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coefficients $a_k$, where k is negative, are different from zero. Addition and multiplication are defined in R<x> as follows:

$$\text{if } g = \sum_{k=-\infty}^{\infty} a_k x^k \text{ and } h = \sum_{k=-\infty}^{\infty} b_k x^k, \text{ then}$$

$$g + h = \sum_{k=-\infty}^{\infty} (a_k + b_k) x^k \text{ and}$$

$$gh = \sum_{k=-\infty}^{\infty} d_k x^k \text{ where } d_k = \sum_{i+j=k} a_i b_j.$$

R<x> is called the <u>field of extended formal power series</u> over R ([4], p. 15).

<u>Theorem 1.5</u>: R<x> is a commutative ring with identity.

Proof: Let f, g, h ε R<x> with

$$f = \sum_{k=-\infty}^{\infty} a_k x^k, \quad g = \sum_{k=-\infty}^{\infty} b_k x^k, \text{ and } h = \sum_{k=-\infty}^{\infty} c_k x^k.$$

$f + g = \sum_{k=-\infty}^{\infty} (a_k + b_k) x^k$ with $a_k + b_k \in R$. Because of the restrictions on f and g, at most a finite number of the coefficients $(a_k + b_k)$, where k is negative, are different from zero. Thus $f + g \in$ R<x>. In a similar manner $fg \in$ R<x>. Hence, R<x> is closed under both operations.

Associativity of addition, commutativity of addition and multiplication, and distributivity are direct consequences of the corresponding properties in R. Associativity of multiplication is just slightly more complex; we present it here.

$$fg = \sum_{s=-\infty}^{\infty} v_s x^s \text{ where } v_s = \sum_{i+j=s} a_i b_j.$$

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$$(fg)h = \sum_{k=-\infty}^{\infty} m_k x^k \quad \text{where } m_k = \sum_{s+r=k} v_s c_r.$$

Then
$$m_k = \sum_{s+r=k} \Big( \sum_{i+j=s} a_i b_j \Big) c_r.$$

On the other hand, $gh = \sum_{t=-\infty}^{\infty} u_t x^t$ where $u_t = \sum_{j+r=t} b_j c_r$.

So
$$f(gh) = \sum_{k=-\infty}^{\infty} w_k x^k \quad \text{where } w_k = \sum_{i+t=k} a_i u_t.$$

Then
$$w_k = \sum_{i+t=k} a_i \Big( \sum_{j+r=t} b_j c_r \Big).$$

Both $m_k$ and $w_k$ involve finite sums because only a finite number of the coefficients with a negative index are non-zero. Therefore,

$$m_k = \sum_{s+r=k} \Big( \sum_{i+j=s} a_i b_j c_r \Big) = \sum_{i+j+r=k} a_i b_j c_r \quad \text{and}$$

$$w_k = \sum_{i+t=k} \Big( \sum_{j+r=t} a_i b_j c_r \Big) = \sum_{i+j+r=k} a_i b_j c_r.$$

Thus $f(gh) = (fg)h$.

1 can be considered to be an element of R<x>, with $a_0 = 1$ and $a_k = 0$ for $k \neq 0$. $1 \cdot g = g$ for all $g \in R\langle x\rangle$. Thus, R<x> is a commutative ring with identity.

A lemma and a theorem will be used to show that R<x> is a field.

Lemma 1.6: If $f = \sum_{k=-\infty}^{\infty} a_k x^k$ where $a_k = 0$ for $k < 0$, and $a_0 \neq 0$, then $f$ has an inverse in R<x>; that is, there

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exists g such that $fg = 1$.

Proof: f can be written as $\sum_{k=0}^{\infty} a_k x^k$.

Let $g = \sum_{k=0}^{\infty} b_k x^k$ (that is $b_k = 0$ for $k < 0$), where $b_k$ for $k = 0, 1, 2, \ldots$, are chosen as follows: We want $a_0 b_0 = 1$, hence let $b_0 = a_0^{-1}$ which exists since $a_0 \neq 0$. We want $a_0 b_1 + a_1 b_0 = 0$, hence let $b_1 = -a_0^{-1} a_1 b_0$. Having chosen $b_0, b_1, \ldots, b_n$, let $b_{n+1}$ be the solution to the equation:

$$a_0 b_{n+1} + a_1 b_n + a_2 b_{n-1} + \cdots + a_n b_1 + a_{n+1} b_0 = 0.$$

Then $b_k \in R$ for all k, and $g \in R\langle x\rangle$. Moreover, $fg = 1$.

<u>Theorem 1.7</u>: $R\langle x\rangle$ is a field.

Proof: $R\langle x\rangle$ has already been shown to be a commutative ring with identity. Let $g \in R\langle x\rangle$ with $g \neq 0$, then $g = \sum_{k=-\infty}^{\infty} a_k x^k$. There must exist an integer r such that $a_r \neq 0$, with $a_k = 0$ for $k < r$. Consider $g \cdot x^{-r}$ which is of the form $\sum_{j=0}^{\infty} b_j x^j$ where $b_j = a_{j+r}$. $b_0 = a_r \neq 0$. By the lemma, $gx^{-r}$ has an inverse h in $R\langle x\rangle$. Therefore $(gx^{-r})h = 1$, implying $g(x^{-r}h) = 1$. Thus g has an inverse in $R\langle x\rangle$, and $R\langle x\rangle$ is a field.

The final step in the sequence is to define an order for $R\langle x\rangle$.

<u>Theorem 1.8</u>: $R\langle x\rangle$ is an ordered field.

Proof: Let P be the subset of $R\langle x\rangle$ consisting of elements $g = \sum_{k=-\infty}^{\infty} a_k x^k$ where the first non-zero coefficient is positive. That is, $g \in P$ if there exists an integer t

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such that $k < t$ implies $a_k = 0$, and $a_t > 0$. Let $h = \sum_{k=-\infty}^{\infty} b_k x^k$ where $k < r$ implies $b_k = 0$, and $b_r > 0$. Let $w = \text{minimum } \{t, r\}$. Then $k < w$ implies $(a_k + b_k) = 0$, and $a_w + b_w > 0$. Thus $g \varepsilon P$, and $h \varepsilon P$, implies $g + h \varepsilon P$.

If $i + j < t + r$, then either $i < t$ or $j < r$. Hence, $k < t + r$ implies $\sum_{i+j=k} a_i b_j = 0$. If $i + j = t + r$ and $i > t$, then $j < r$; if $i + j = t + r$ and $j > r$, then $i < t$. Hence, $k = t + r$ implies that $\sum_{i+j=k} a_i b_j = a_t b_r > 0$. Thus $g \varepsilon P$, and $h \varepsilon P$, implies $gh \varepsilon P$.

Clearly $g \varepsilon R\langle x\rangle$ implies that $g \varepsilon P$, $g = 0$, or $-g \varepsilon P$.

Therefore, P forms a positive class for R<x>, and R<x> is an ordered field.

R<x> will serve as an important counter-example in the next chapter.

The next item to be discussed is the imbedding of the field of rational numbers in any ordered field. Let F be an ordered field, let $Z' = \{me \mid m \varepsilon Z\}$, and let $Q' = \{(me)(ne)^{-1} \mid m \varepsilon Z, n \varepsilon N\}$. Since $me = me \circ e = me \circ e^{-1} = me \circ (1e)^{-1}$, we have $Z' \subseteq Q' \subseteq F$.

<u>Theorem 1.9</u>: Let F be an ordered field and let $\psi : Q \to F$ be defined by $\psi(m/n) = (me)(ne)^{-1}$. Then $\psi$ is an order-preserving field isomorphism of Q into F, $\psi(Q) = Q'$, and $\psi(Z) = Z'$.

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Proof: Suppose $m/n = r/s$. Then $ms = nr$ which implies $(ms)e = (nr)e$. Hence $(me)(se) = (ne)(re)$ and $(me)(ne)^{-1} = (re)(se)^{-1}$. Thus $\psi(m/n) = \psi(r/s)$ and $\psi$ is well defined.

Suppose $\psi(m/n) = \psi(r/s)$. Then $(me)(ne)^{-1} = (re)(se)^{-1}$ which implies $(me)(se) = (re)(ne)$. Hence $(ms)e = (rn)e$ and $ms = rn$. Thus $m/n = r/s$ and $\psi$ is one-to-one.

$$
\begin{aligned}
\psi(m/n + r/s) &= \psi((ms + nr)/(ns)) \\
&= ((ms + nr)e)((ns)e)^{-1} \\
&= (mse + nre)(nse)^{-1} \\
&= (mse)(nse)^{-1} + (nre)(nse)^{-1} \\
&= (me)(se)(ne)^{-1}(se)^{-1} \\
&\quad + (ne)(re)(ne)^{-1}(se)^{-1} \\
&= (me)(ne)^{-1} + (re)(se)^{-1} \\
&= \psi(m/n) + \psi(r/s)
\end{aligned}
$$

$$
\begin{aligned}
\psi((m/n)(r/s)) = \psi((mr)/(ns)) &= (mre)(nse)^{-1} \\
&= ((me)(re))((ne)(se))^{-1} \\
&= (me)(re)(ne)^{-1}(se)^{-1} \\
&= (me)(ne)^{-1}(re)(se)^{-1} = \psi(m/n)\psi(r/s)
\end{aligned}
$$

Therefore, $\psi$ is a field isomorphism.

$m/n \in Q$ implies $\psi(m/n) = (me)(ne)^{-1} \in Q'$; thus $\psi(Q) \subseteq Q'$. Moreover, every element of $Q'$ is of the form $(re)(se)^{-1}$ where $r \in Z$, and $s \in N$. $r/s \in Q$ and $\psi(r/s) = (re)(se)^{-1}$. Thus $Q' \subseteq \psi(Q)$. Therefore $\psi(Q) = Q'$.

Let $m \in Z$, then $\psi(m) = \psi(m/1) = (me)(1e)^{-1} = (me)(e)^{-1} = (me)(e) = me \in Z'$, hence $\psi(Z) \subseteq Z'$. Let $re \in Z'$; then $re = (re)(1e)^{-1} = \psi(r/1) = \psi(r)$ where $r \in Z$.

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Thus, $Z' \subseteq \psi(Z)$ and therefore $\psi(Z) = Z'$.

Let $m/n,\ p/q \in Q$. We may assume that $n,\ q \in N$. $m/n < p/q$ implies $mq < np$ which, by lemma 1.3, implies $(mq)e < (np)e$, and this in turn implies $(me)(qe) < (ne)(pe)$. Also by lemma 1.3, $(ne)^{-1} > \Theta$ and $(qe)^{-1} > \Theta$. Thus $m/n < p/q$ implies $(ne)^{-1}(qe)^{-1}(me)(qe) < (ne)^{-1}(qe)^{-1}(ne)(pe)$, which implies $(me)(ne)^{-1} < (pe)(qe)^{-1}$, resulting in $\psi(m/n) < \psi(p/q)$. Therefore, $\psi$ is order-preserving and the proof is finished.

Since Z is order-isomorphic to Z' and Q is order-isomorphic to Q', from this point on we assume, for any ordered field F, that $N \subseteq Z \subseteq Q \subseteq F$. Consequently, 0 will be used instead of $\Theta$, 1 will be used instead of e, and $m/n$ will be used instead of $(me)(ne)^{-1}$. The elements of Q will be called <u>rational elements</u> of F, the elements of $F - Q$, if any, will be called <u>irrational elements</u> of F. The <u>well-ordering</u> property for the set N of natural numbers says that every non-empty subset of N has a least element. This property also holds when N is considered to be a subset of F. We will sometimes use $x/y$ for $x \cdot y^{-1}$ where $x,\ y \in F$. Making immediate use of this, we note that $a,\ b \in F$ with $a < b$, implies $a < (a + b)/2 < b$.

<u>Definition 1.10</u>: An ordered field F is said to be an <u>Archimedean ordered field</u> if for each $x \in F$, with $x > 0$, there exists $n \in N$ such that $x < n$.

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Essentially, F is Archimedean if N is "unbounded" in F. Of the previous examples of ordered fields, R and Q are Archimedean. Since Q may be considered a subfield of any ordered field, every ordered field has an Archimedean subfield. Not all ordered fields are Archimedean. To see that Q(x) is non-Archimedean, observe that $n - x < 0$ for all $n \varepsilon N$. Thus there is no $n \varepsilon N$ such that $x < n$. In a similar manner, using $x^{-1}$, one can show that R<x> is also not Archimedean. It will be seen that Archimedean ordering plays an important role in the next chapter. Some of the basic properties of an Archimedean ordered field are presented in the next lemma.

<u>Lemma 1.11</u>: If F is an Archimedean ordered field and $y, z \varepsilon F$ then:

(i) $y > 0$ and $z > 0$ implies there exists $n \varepsilon N$ such that $ny > z$.

(ii) $z > 0$ implies there exists $n \varepsilon N$ such that $0 < 1/n < z$.

(iii) $y > 0$ implies there exists $n \varepsilon N$ such that $n - 1 \leq y < n$.

(iv) If $y < z$ then there exists a rational element $m/n$ such that $y < m/n < z$.

Proof: (i) If $y > 0$ then $y^{-1} > 0$. Thus $zy^{-1} > 0$ and there exists $n \varepsilon N$ such that $n > zy^{-1}$. Hence, $ny > z$.

(ii) There exists $n \varepsilon N$ such that $nz > 1$. Hence

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$z > 1/n > 0$.

(iii) There exists $n \in N$ such that $y < n$. As N is well-ordered one may choose n to be the smallest such integer. Then $n - 1 \leq y < n$.

(iv) There will be no loss of generality in assuming that $0 < y < z$ as the proofs for the other cases would be based on this one. $z - y > 0$ so there exists $n \in N$ with $1/n < z - y$. Choose $m \in N$ such that $m - 1 \leq ny < m$. Then $m/n \leq y + 1/n$, while $y < m/n$ and $y + 1/n < z$. Therefore $y < m/n < z$.

Before considering the structure of an Archimedean ordered field, we introduce the concepts of interval, absolute value, and sequences.

<u>Definition 1.12</u>: Let F be an ordered field. For $a, b \in F$ we define:

(i) $]a, b[ = \{x | x \in F \text{ and } a < x < b\}$ if $a < b$, and call $]a, b[$ an <u>open interval</u> of F;

(ii) $[a, b] = \{x | x \in F \text{ and } a \leq x \leq b\}$ if $a \leq b$, and call $[a, b]$ a <u>closed interval</u> of F.

<u>Definition 1.13</u>: If F is an ordered field and $x \in F$, then the <u>absolute value</u> of x, denoted by $|x|$, is defined by:

$$|x| = \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x < 0. \end{cases}$$

These definitions parallel those given for the real numbers as do the properties listed in the next lemma.

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The proof of the lemma would be constructed the same as that given in elementary calculus and is omitted here.

<u>Lemma 1.14</u>: Let F be an ordered field and x, y $\varepsilon$ F, then:

(i) $|x| \geq 0$.

(ii) $|x| = 0$ if and only if $x = 0$.

(iii) $|x - y| = |y - x|$.

(iv) $|xy| = |x| \cdot |y|$.

(v) $|x/y| = |x|/|y|$, $y \neq 0$.

(vi) If a $\varepsilon$ F with $a > 0$ then $|x| < a$ if and only if $-a < x < a$; and $|x| \leq a$ if and only if $-a \leq x \leq a$.

(vii) $|x| - |y| \leq |x \pm y| \leq |x| + |y|$.

(viii) $||x| - |y|| \leq |x - y|$.

The definitions 1.15 through 1.21 involve sequences.

<u>Definition 1.15</u>: A <u>sequence</u> in an ordered field F is a function X whose domain is the set N and whose range is contained in F. The value of X at n, X(n), will be denoted by $x_n$. The sequence will be denoted by X, $\{x_n\}$, or $(x_n)_{n=1}^{\infty}$.

<u>Definition 1.16</u>: Let X be a sequence in an ordered field F. Let $n_1 < n_2 < n_3 < \ldots < n_k < n_{k+1} < \ldots$ be any strictly increasing sequence of positive integers. Then $(x_{n_k})_{k=1}^{\infty}$ is a sequence and is called a <u>subsequence</u> of X.

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Definition 1.17: A sequence $\{x_n\}$ in an ordered field F is said to converge to an element x of F, if for every element $\mu \in F$ such that $\mu > 0$, there exists an $M \in N$ such that $n \geq M$ implies $|x - x_n| < \mu$. x is said to be the limit of the sequence $\{x_n\}$ and one writes $\lim_{n \to \infty} x_n = x$.

Definition 1.18: A sequence $\{x_n\}$ in an ordered field F is said to be a Cauchy Sequence in F, if for every element $\mu \in F$ such that $\mu > 0$, there exists an $M \in N$ such that $n \geq M$ and $m \geq M$ implies $|x_n - x_m| < \mu$.

Definition 1.19: An ordered field F is said to be Cauchy Complete if every Cauchy sequence in F converges to an element of F.

As noted in the introduction, R is Cauchy Complete while Q is not. We show in chapter two that R<x> is Cauchy Complete.

Definition 1.20: A sequence $\{x_n\}$ in an ordered field F is said to be bounded if there exists $t \in F$ such that $|x_n| \leq t$ for all n.

Definition 1.21: The sequence $\{x_n\}$ in an ordered field F is said to be monotone increasing (decreasing) if for all $n \in N$, $x_n \leq x_{n+1}$ $(x_{n+1} \leq x_n)$. A sequence is said to be monotone if it is either monotone increasing or monotone decreasing.

The next lemma is a compilation of some of the important consequences of these definitions.

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<u>Lemma 1.22</u>: Let $\{x_n\}$ and $\{y_n\}$ be convergent sequences in an ordered field F with $\lim_{n \to \infty} x_n = x$ and $\lim_{n \to \infty} y_n = y$. Then:

(i) $\lim_{n \to \infty} x_n$ is unique.

(ii) If $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$, then $\{x_{n_k}\}$ converges and $\lim_{k \to \infty} x_{n_k} = x$.

(iii) $\{x_n\}$ is a bounded sequence.

(iv) $\{x_n \pm y_n\}$ is a convergent sequence and $\lim_{n \to \infty} (x_n \pm y_n) = x \pm y$.

(v) $\{x_n y_n\}$ is a convergent sequence and $\lim_{n \to \infty} (x_n y_n) = xy$.

(vi) If $y_n \neq 0$ for all n, and $y \neq 0$, then $\{x_n/y_n\}$ is a convergent sequence and $\lim_{n \to \infty} (x_n/y_n) = x/y$.

(vii) For all $k \in F$, $\{kx_n\}$ converges and $\lim_{n \to \infty} kx_n = kx$.

Proof: (i) Assume that it is also true that $\lim_{n \to \infty} x_n = z$. Let $\mu \in F$ with $\mu > 0$, then there exist $M_1$, $M_2 \in N$ such that $n \geq M_1$ implies $|x_n - x| < \mu/2$, and $n \geq M_2$ implies $|x_n - z| < \mu/2$. Let M be the maximum of $M_1$ and $M_2$. Then $n \geq M$ implies

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$$|x - z| \leq |x_n - x| + |z - x_n| < \mu.$$

But $\mu$ was arbitrary; so $|x - z| = 0$ and $x = z$.

(ii) Let $\mu \in F$ with $\mu > 0$, then there exists $M \in N$ such that $n \geq M$ implies $|x_n - x| < \mu$. $\{n_k\}$ is a strictly increasing sequence in N, hence $n_k \geq k$ for each k. Thus $k \geq M$ implies $|x_{n_k} - x| < \mu$. Therefore, $\lim_{k \to \infty} x_{n_k} = x$.

(iii) There exists $K \in N$ with $n \geq K$ implying $|x_n - x| < 1$. But $|x_n| - |x| \leq |x_n - x|$ means $n \geq K$ implies $|x_n| < |x| + 1$. Let $L = \text{maximum } \{|x_1|, |x_2|, |x_3|, \ldots, |x_{K-1}|, |x| + 1\}$. For all n, $|x_n| \leq L$, so $\{x_n\}$ is a bounded sequence.

(iv) Let $\mu \in F$ with $\mu > 0$, then there exists $M \in N$ such that $n \geq M$ implies $|x_n - x| < \mu/2$ and $|y_n - y| < \mu/2$. Thus $n \geq M$ implies $|(x_n + y_n) - (x + y)| < \mu$. Therefore, $\lim_{n \to \infty} (x_n + y_n) = x + y$. It may be shown in a similar manner that $\lim_{n \to \infty} (x_n - y_n) = x - y$.

(v)
$$|x_n y_n - xy| = |(x_n y_n - x y_n) + (x y_n - xy)| \leq |x_n - x|\,|y_n| + |y_n - y|\,|x|.$$

$\{y_n\}$ is bounded so L may be chosen such that $|y_n| \leq L$ for all n and also so that $|x| \leq L$. If $y_n = 0$ for all n, then $y = 0$ and $\lim_{n \to \infty} (x_n y_n) = xy$. If not, then $L > 0$. Let $\mu \in F$ with $\mu > 0$, then choose M so that $n \geq M$ implies both $|x_n - x| < \mu/2L$ and $|y_n - y| < \mu/2L$. Then $n \geq M$ implies

$$|x_n y_n - xy| < \frac{\mu |y_n|}{2L} + \frac{\mu |x|}{2L}.$$

Hence $|x_n y_n - xy| < \mu$ for $n \geq M$, and $\lim_{n \to \infty} (x_n y_n) = xy$.

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(vi) This will follow directly from (v) once one has shown that $\lim_{n \to \infty} (1/y_n) = 1/y$. $y \neq 0$ means $|y|/2 > 0$. hence there exists $M_1$ such that $n \geq M_1$ implies $|y - y_n| < |y|/2$. Recalling that $|y| - |y_n| \leq |y - y_n|$, we conclude that $n \geq M_1$ implies $|y|/2 < |y_n|$. Let $\mu \in F$ with $\mu > 0$, then there exists $M_2 \in N$ such that $n \geq M_2$ implies $|y_n - y| < \mu|y|^2/2$. Let M be the maximum of $M_1$ and $M_2$. Then $n \geq M$ implies

$$|1/y_n - 1/y| = \frac{|y - y_n|}{|y_n||y|} \leq \frac{2|y - y_n|}{|y|^2} .$$

Thus $n \geq M$ implies $|1/y_n - 1/y| < \mu$. Hence, $\lim_{n \to \infty} (1/y_n) = 1/y$.

(vii) This is a direct consequence of (v) with $y_n = k$ for all n.

Every Archimedean ordered field may be considered a subfield of R. The next lemmas lay the groundwork for the proof of this assertion.

Lemma 1.23: If F is an Archimedean ordered field, then for all $a \in F$ there exists a sequence $\{r_n\}$ in Q such that $\{r_n\}$ converges to a.

Proof: By lemma 1.11, for each $n \in N$, there exists $r_n \in Q$ such that $a < r_n < a + 1/n$. Let $\mu \in F$ with $\mu > 0$. Then there exists $M \in N$ such that $1/M < \mu$. For $n \geq M$, $1/n \leq 1/M$. Hence for $n \geq M$, $a < r_n < a + 1/n \leq a + 1/M$. Thus $n \geq M$ implies $|r_n - a| = r_n - a < 1/M < \mu$. That is, $\lim_{n \to \infty} r_n = a$.

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Lemma 1.24: If F is an ordered field and $\{x_n\}$ converges to x in F, then $\{x_n\}$ is a Cauchy sequence in F.

Proof: Let $\mu \varepsilon F$ with $\mu > 0$, then there exists $M \varepsilon N$ such that $n \geq M$ implies $|x_n - x| < \mu/2$. Since $|x_n - x_m| \leq |x_n - x| + |x - x_m|$; n, $m \geq M$ implies that $|x_n - x_m| < \mu$. Hence, $\{x_n\}$ is a Cauchy sequence.

Lemma 1.25: If F is an ordered field and $\{r_n\}$ is a Cauchy sequence in F where $r_n \varepsilon Q$ for all n, then $\{r_n\}$ is a Cauchy sequence in R and hence $\{r_n\}$ converges in R.

Proof: Let $\mu \varepsilon R$ with $\mu > 0$. R is Archimedean; so there exists $M \varepsilon N$ such that $0 < 1/M < \mu$. But $1/M \varepsilon Q$, hence $1/M \varepsilon F$. There exists $K \varepsilon N$ such that n, $m \geq K$ implies $|r_n - r_m| < 1/M$ in F. Thus n, $m \geq K$ implies $|r_n - r_m| < 1/M$ in Q, considered as a subset of R. Consequently, n, $m \geq K$ implies $|r_n - r_m| < \mu$ in R. Thus $\{r_n\}$ is a Cauchy sequence in R, and as R is Cauchy Complete, $\{r_n\}$ converges in R.

Lemma 1.26: Let F be an Archimedean ordered field and $\{r_n\}$ be a sequence with $r_n \varepsilon Q$ for all n. Then $\{r_n\}$ converges to 0 in F if and only if $\{r_n\}$ converges to 0 in R.

Proof: Suppose $\{r_n\}$ converges to 0 in F. Let $\mu \varepsilon R$ with $\mu > 0$. Since R is Archimedean, there exists $M \varepsilon N$ such that $0 < 1/M < \mu$. $1/M \varepsilon Q$ implies $1/M \varepsilon F$. Hence there exists $K \varepsilon N$ such that $n \geq K$ implies

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$|r_n| < 1/M$ in F. As $r_n \in Q$ for all n, $n \geq K$ implies $|r_n| < 1/M$ in R, which in turn implies $|r_n| < \mu$ in R. Thus $\{r_n\}$ converges to 0 in R.

The converse is shown similarly since F is Archimedean and $Q \subseteq F$.

<u>Lemma 1.27</u>: Let F be an Archimedean ordered field. Let $\{r_n\}$ and $\{s_n\}$ be sequences in Q. Let a, b $\in$ F with $a < b$, and suppose that in F, $\lim_{n \to \infty} r_n = a$ and $\lim_{n \to \infty} s_n = b$. Then in R, $\lim_{n \to \infty} r_n < \lim_{n \to \infty} s_n$.

Proof: By lemmas 1.24 and 1.25, $\{r_n\}$ and $\{s_n\}$ converge in F implies $\{r_n\}$ and $\{s_n\}$ converge in R. In F, $a < b$ implies $(b - a)/4 > 0$. Choose $M \in N$ such that $n \geq M$ implies both $|r_n - a| < (b - a)/4$ and $|s_n - b| < (b - a)/4$. Then $n \geq M$ implies $r_n < a + (b - a)/4$ and $b - (b - a)/4 < s_n$. But $a + (b - a)/4 < (a + b)/2 < b - (b - a)/4$, hence by lemma 1.11 there exist p, q $\in$ Q such that

$$a + (b - a)/4 < p < (a + b)/2 < q < b - (b - a)/4.$$

Thus, $n \geq M$ implies $r_n < p < q < s_n$ in F. As a result, $n \geq M$ implies $r_n < p < q < s_n$ in R. Thus in R, $\lim_{n \to \infty} r_n \leq p < q \leq \lim_{n \to \infty} s_n$. That is, $\lim_{n \to \infty} r_n < \lim_{n \to \infty} s_n$.

Lemmas 1.23-1.27 are used in the proof of the next theorem which is a significant result.

<u>Theorem 1.28</u>: Let F be an Archimedean ordered

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field. Then there exists $\varphi: F \longrightarrow R$ such that $\varphi$ is an order-preserving field isomorphism of F into R.

Proof: Let $a \in F$, then there exists a sequence $\{r_n\}$ in Q which converges to a in F. By lemmas 1.24 and 1.25, $\{r_n\}$ also converges in R. Let $\lim_{n \to \infty} r_n = a'$ in R. Define $\varphi: F \longrightarrow R$ by $\varphi(a) = a'$.

Suppose $\{r_n\}$ converges to a in F and suppose $\{s_n\}$ also converges to a in F. Then $\{r_n - s_n\}$ converges to 0 in F. Hence $\{r_n - s_n\}$ converges to 0 in R, by lemma 1.26. Thus, $\lim_{n \to \infty} r_n = \lim_{n \to \infty} s_n$ in R, as well as F, and $\varphi$ is well-defined.

Suppose that in F, $\{r_n\}$ converges to a and $\{s_n\}$ converges to b. Then $\{r_n + s_n\}$ converges to a + b. Thus in R, $\varphi(a) + \varphi(b) = \lim_{n \to \infty} r_n + \lim_{n \to \infty} s_n = \lim_{n \to \infty} (r_n + s_n)$

$$= \varphi(a + b)$$

Also, $\{r_n s_n\}$ converges to ab in F. Thus in R,
$\varphi(a) \cdot \varphi(b) = (\lim_{n \to \infty} r_n)(\lim_{n \to \infty} s_n) = \lim_{n \to \infty} (r_n s_n) = \varphi(ab)$.
Therefore, $\varphi$ is a field homorphism.

Suppose $\varphi(a) = \varphi(b)$. Let $\lim_{n \to \infty} r_n = a$ and $\lim_{n \to \infty} s_n = b$. Then $\lim_{n \to \infty} (r_n - s_n) = \varphi(a) - \varphi(b) = 0$ in R which, by lemma 1.26, implies $\lim_{n \to \infty} (r_n - s_n) = 0$ in F. That means a = b. Hence $\varphi$ is one-to-one, and an isomorphism.

Finally, lemma 1.27 says that $\varphi$ is order-preserving.

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A topology for an ordered field will be the last item to be discussed in this chapter. Let F be an ordered field. Let $B = \{]a, b[ \mid a < b \text{ in } F\}$. For any $x \in F$, there exist $a, b \in F$ such that $a < x < b$, and thus $F = \bigcup \{I \mid I \in B\}$. Note also that if $I, J \in B$, then either $I \cap J \in B$ or $I \cap J = \emptyset$. Let $\mathcal{T}$ be the family of all unions of members of B. A union of members of $\mathcal{T}$ is itself a union of members of B and therefore is a member of $\mathcal{T}$. Moreover, if U and V are members of $\mathcal{T}$ and if $x \in U \cap V$, then there exist $I, J \in B$ such that $x \in I \subseteq U$ and $x \in J \subseteq V$. Thus $x \in I \cap J \subseteq U \cap V$ and it follows that $U \cap V$ is a union of members of B. Therefore $\mathcal{T}$ is closed under the formation of arbitrary unions and finite intersections. The family $\mathcal{T}$ is called a <u>topology</u> for F and the family B is called a <u>base</u> for the topology $\mathcal{T}$. The members of $\mathcal{T}$ will be called <u>open sets</u> of F and set complements of members of $\mathcal{T}$ will be called <u>closed sets</u> of F.

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# CHAPTER II

## AN INVESTIGATION OF COMPLETENESS

As noted in the introduction, we assume that the field of real numbers is a complete Archimedean ordered field. The definition of complete depends on which text is consulted. Most authors of Advanced Calculus texts presuppose the existence of the real numbers and take one of several possible characterizations of completeness as an axiom. A detailed description of the construction of R on the basis of Dedekind cuts is given in Rudin [6].

Working in the general context of an ordered field, we investigate eight properties pertaining to completeness, showing that six of the eight properties are equivalent, and that each of the six implies the other two. Cauchy Completeness is one of the properties. The others are introduced in definitions 2.3, and 2.7-2.12. We show that each of the six equivalent properties implies that the ordering is Archimedean and that in an Archimedean ordered field all eight completeness properties are equivalent.

We conclude with an inquiry into Archimedean ordered fields which are "complete". We show that there is essentially only one such field, the field of real numbers. That is, we show that any complete Archimedean ordered field is order-isomorphic to R.

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Definition 2.1: Let S be a non-empty subset of an ordered field F. An element u of F is said to be an upper bound of S if $x \leq u$ for all $x \in S$. An element v of F is said to be a lower bound of S if $v \leq x$ for all $x \in S$.

If a set S has an upper bound, it is said to be bounded above; if S has a lower bound, it is said to be bounded below. If S is bounded both above and below, S is said to be bounded. S is said to be unbounded if it either has no upper bound or has no lower bound.

Definition 2.2: Let S be a non-empty subset of an ordered field F. An element u of F is called a least upper bound of S if u is an upper bound of S, and $u \leq w$ for all upper bounds w of S. An element v of F is called a greatest lower bound of S if v is a lower bound of S, and $t \leq v$ for all lower bounds t of S.

It is easily shown that if S has a least upper bound, or greatest lower bound, then it is unique. The following is a property used very often to characterize the real numbers.

Definition 2.3: An ordered field F is said to be Least Upper Bound Complete if every non-empty subset of F, which has an upper bound in F, has a least upper bound in F.

The next theorem is an immediate consequence of this definition.

Theorem 2.4: If F is an ordered field which is

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Least Upper Bound Complete, then every non-empty subset of $F$ which has a lower bound in $F$ has a greatest lower bound in $F$.

Proof: Let $S$ be a non-empty subset of $F$ and let $v$ be a lower bound of $S$. Let $T = \{y \mid -y \in S\}$, then $T \neq \emptyset$. If $y \in T$, then $-y \in S$, which implies $v \leq -y$, and hence $y \leq -v$. Thus $-v$ is an upper bound for $T$ and $T$ has a least upper bound $w$. If $r \in S$, then $-r \in T$, which implies $-r \leq w$ and in turn $-w \leq r$. Thus $-w$ is a lower bound for $S$. Let $x$ be any lower bound for $S$, then by the argument above, $-x$ is an upper bound for $T$. Hence, $w \leq -x$ and $x \leq -w$. Therefore, $-w$ is the greatest lower bound of $S$.

Definition 2.5: Let $J$ be a non-empty subset of an ordered field $F$. An element $x \in F$ is said to be a limit point of $J$ if for every $\mu \in F$ with $\mu > 0$, there exists $y \in J$ such that $0 < |x - y| < \mu$.

A limit point is sometimes called a "cluster point" or an "accumulation point". It should be noted that $x$ is a limit point of $J$ if and only if for every $\mu \in F$ with $\mu > 0$, there exists $y \in F$ with $y \neq x$, such that $y \in I \cap J$, where $I = ]x - \mu, x + \mu[$.

Definition 2.6: In an ordered field $F$ the family $\{]c_\alpha, d_\alpha[ \mid \alpha \in A\}$ of open intervals is said to be a covering by open intervals of the closed interval $[a, b]$, if for each element $x \in [a, b]$, there is $\alpha \in A$ such that

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$x \in \,]c_\alpha, d_\alpha[$. That is to say, $[a, b] \subseteq \bigcup_{\alpha \in A} \,]c_\alpha, d_\alpha[$.

We now list, in definitions 2.7-2.12, the remaining properties which describe the various notions of completeness.

Definition 2.7: An ordered field F is said to satisfy the Bolzano-Weierstrass Set Property if every bounded infinite subset of F has a limit point.

Definition 2.8: An ordered field F is said to satisfy the Bolzano-Weierstrass Sequence Property if every bounded sequence in F has a convergent subsequence.

Definition 2.9: An ordered field F is said to satisfy the Bounded Monotone Sequence Property if every bounded monotone sequence in F is convergent.

Definition 2.10: An ordered field F is said to be disconnected if there exist open sets A and B of F such that $A \neq \emptyset$, $B \neq \emptyset$, $A \cap B = \emptyset$, and $A \cup B = F$. If F is not disconnected, then F is said to be connected.

Definition 2.11: An ordered field F is said to satisfy the Heine-Borel Covering Property if each covering by open intervals of a closed and bounded interval contains a finite subcovering of that interval. That is, if $\{I_\alpha \mid \alpha \in A\}$ is a collection of open intervals of F and $[a, b] \subseteq \bigcup_{\alpha \in A} I_\alpha$, then there exist $I_{\alpha_j}$ for $j = 1, 2, \ldots, n$ such that $[a, b] \subseteq \bigcup_{j=1}^{n} I_{\alpha_j}$.

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<u>Definition 2.12</u>: An ordered field F is said to satisfy the <u>Nested Interval Property</u> if when $\{I_k\}$ is a sequence of closed and bounded intervals in F which is nested, in the sense that $I_1 \supseteq I_2 \supseteq \ldots \supseteq I_k \supseteq I_{k+1} \supseteq \ldots$, then the intersection of the intervals $I_k$ is non-empty.

The six properties specified in definitions 2.3 and 2.7-2.11 are equivalent and imply that the ordering is Archimedean. Before showing this, we take a look at a few examples.

A verification that R is Least Upper Bound Complete may be found on page 11 of Rudin's book [6]. That R satisfies the other seven properties as well will become apparent later in this chapter. The field Q of rational numbers is a standard example of a field which is neither Least Upper Bound Complete nor Cauchy Complete.

R<x>, introduced previously, is an example of an ordered field which is Cauchy Complete but not Least Upper Bound Complete. We will show that every ordered field which is Least Upper Bound Complete is Archimedean, hence, R<x> can not be Least Upper Bound Complete. But first we proceed to show that R<x> is Cauchy Complete. The following lemma will be needed.

<u>Lemma 2.13</u>: If $\sum_{k=-\infty}^{\infty} d_k x^k$ is an element of R<x>, then $\sum_{k=-\infty}^{\infty} |d_k| x^k$ is also in R<x>, and
$|\sum_{k=-\infty}^{\infty} d_k x^k| \leq \sum_{k=-\infty}^{\infty} |d_k| x^k$.

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Proof: Clearly, if $\sum_{k=-\infty}^{\infty} d_k x^k \in R\langle x\rangle$, then $\sum_{k=-\infty}^{\infty} |d_k| x^k \in R\langle x\rangle$.

If $d_k = 0$ for all k, the lemma obviously holds. So we may assume $d_k \neq 0$ for some k. Let r be such that $k < r$ implies $d_k = 0$ and $d_r \neq 0$.

If $d_r > 0$, then $|\sum_{k=-\infty}^{\infty} d_k x^k| = \sum_{k=-\infty}^{\infty} d_k x^k$. $\sum_{k=-\infty}^{\infty} |d_k| x^k - \sum_{k=-\infty}^{\infty} d_k x^k = \sum_{k=-\infty}^{\infty} (|d_k| - d_k) x^k$. Since, in R, $d_k \leq |d_k|$, and hence $|d_k| - d_k \geq 0$, we have $\sum_{k=-\infty}^{\infty} |d_k| x^k - \sum_{k=-\infty}^{\infty} d_k x^k \geq 0$, and the desired inequality holds.

If $d_r < 0$, then $|\sum_{k=-\infty}^{\infty} d_k x^k| = \sum_{k=-\infty}^{\infty} -d_k x^k$. Moreover, $\sum_{k=-\infty}^{\infty} |d_k| x^k - \sum_{k=-\infty}^{\infty} -d_k x^k = \sum_{k=-\infty}^{\infty} (|d_k| + d_k) x^k$. Once again the inequality holds, since in R, $-d_k \leq |d_k|$, and hence $|d_k| + d_k \geq 0$.

<u>Theorem 2.14</u>: R<x> is Cauchy Complete.

Proof: Let $\{y_n\}$ be a Cauchy sequence in R<x>, with $y_n = \sum_{k=-\infty}^{\infty} a_{n,k} x^k$.

Let $\delta \in R$ with $\delta > 0$ and for each integer k, let $\delta_k = \delta x^k$ in R<x>. For each integer k, since $\delta_k > 0$ and $\{y_n\}$ is a Cauchy sequence in R<x>, there exists an $M_k \in N$ such that $n, m \geq M_k$ implies $|y_n - y_m| < \delta_k$. This means that for $n, m \geq M_k$ and $j < k$, $|a_{n,j} - a_{m,j}| = 0$

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and $|a_{n,k} - a_{m,k}| < \delta$.

$\delta$ was an arbitrary positive element of R; so for each integer k, $\{a_{n,k}\}$ is a Cauchy sequence in R. Since R is Cauchy Complete, for each k there exists $b_k \in R$, such that $\lim_{n\to\infty} a_{n,k} = b_k$. Let $b = \sum_{k=-\infty}^{\infty} b_k x^k$. We show that $b \in R\langle x\rangle$ and $\lim_{n\to\infty} y_n = b$.

Since for a given k, there exists $M_k$ such that $n, m \geq M_k$ and $j < k$ imply $|a_{n,j} - a_{m,j}| = 0$, $n \geq M_k$ and $j < k$ imply $a_{n,j} = a_{M_k,j}$. $y_{M_k} \in R\langle x\rangle$ implies there exists an r such that for $j < r$, $a_{M_k,j} = 0$. Let t be the minimum of r and k, then $j < t$ and $n \geq M_k$ imply $a_{n,j} = 0$. Thus for $j < t$, $b_j = 0$. Hence $b \in R\langle x\rangle$.

Given $\mu \in R\langle x\rangle$, with $\mu > 0$, there exists an integer s such that $\mu = \sum_{k=-\infty}^{\infty} \mu_k x^k$ where $\mu_k = 0$ for $k < s$ and $\mu_s > 0$. As noted previously, there exists $M_s$ such that $n \geq M_s$ and $k < s$ imply $a_{n,k} = a_{M_s,k}$. But $\lim_{n\to\infty} a_{n,k} = b_k$, so $n \geq M_s$ and $k < s$ imply $b_k = a_{n,k} = a_{M_s,k}$. Moreover, since $\lim_{n\to\infty} a_{n,s} = b_s$, and $\mu_s > 0$, there exists K such that $n \geq K$ implies $|a_{n,s} - b_s| < \mu_s$.

$$\mu - \sum_{k=-\infty}^{\infty} |a_{n,k} - b_k| x^k = \sum_{k=-\infty}^{\infty} (\mu_k - |a_{n,k} - b_k|) x^k.$$

Let L be the maximum of $M_s$ and K. If $n \geq L$ and $k < s$, then $\mu_k = 0$; $a_{n,k} = b_k$, that is, $|a_{n,k} - b_k| = 0$; and $\mu_s - |a_{n,s} - b_s| > 0$. Thus for $n \geq L$, the first non-zero

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coefficient in $\sum_{k=-\infty}^{\infty}(\mu_k - |a_{n,k} - b_k|)x^k$ occurs at k = s and is positive in R. Hence, the expression itself is positive in R<x>. That is, $n \geq L$ implies $\sum_{k=-\infty}^{\infty}|a_{n,k} - b_k x^k| < \mu$. By lemma 2.13,

$$|y_n - b| = |\sum_{k=-\infty}^{\infty}(a_{n,k} - b_k)x^k| \leq \sum_{k=-\infty}^{\infty}|a_{n,k} - b_k|x^k < \mu,$$

for $n \geq L$. Thus $\{y_n\}$ converges to b, and R<x> is Cauchy Complete.

Upon investigating the relationships between the various types of completeness, we find that our investigation bears little fruit if the ordering is not Archimedean. In theorems 2.17-2.22 we show that most of the notions of completeness do indeed imply that the field is Archimedean ordered. We precede these with an alternate criterion for Archimedean ordering.

<u>Theorem 2.15</u>: An ordered field F is Archimedean if and only if the sequence $\{1/n\}$ converges to zero.

Proof: Suppose F is Archimedean. Let $\mu \in F$ with $\mu > 0$. Then by lemma 1.11, there exists $M \in N$ such that $1/M < \mu$. If $n \geq M$, then $1/n \leq 1/M$; so $n \geq M$ implies $|1/n - 0| = 1/n < \mu$. Thus $\{1/n\}$ converges to zero.

Suppose $\{1/n\}$ converges to zero. If $x \in F$ with $x > 0$, then $1/x > 0$, and there exists $M \in N$, such that $n \geq M$ implies $1/n < 1/x$. In particular, $1/M < 1/x$; so $x < M$ and F is Archimedean.

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Corollary 2.16: If $\{a_n\}$ is a strictly increasing sequence of positive integers in an Archimedean ordered field F, then $\{1/a_n\}$ converges to zero.

Proof: $\{1/a_n\}$ is a subsequence of $\{1/n\}$.

Theorem 2.17: If F is an ordered field which is Least Upper Bound Complete, then F is Archimedean ordered, and hence the set N has no upper bound in F.

Proof: Assume F is not Archimedean ordered. Then there exists $x \in F$ such that $n \leq x$ for all $n \in N$. x is an upper bound for N so N has a least upper bound in F; call it t. $n \leq t$ for all $n \in N$. But $n + 1 \in N$ for all $n \in N$; so $n + 1 \leq t$ for all $n \in N$, and $n \leq t - 1$ for all $n \in N$. That is, $t - 1$ is an upper bound for N; but $t - 1 < t$ and this is contrary to t being the least upper bound for N. Therefore, F is Archimedean ordered.

Theorem 2.18: If F is an ordered field which satisfies the Bounded Monotone Sequence Property, then F is Archimedean ordered.

Proof: Assume F is not Archimedean ordered. Then there exists $x \in F$ such that $n \leq x$ for all $n \in N$. Thus the sequence $\{n\}$ is a bounded monotone sequence and converges, say to t. Note that if $\{y_n\}$ is a monotone increasing sequence and $\lim_{n \to \infty} y_n = s$, then $y_n \leq s$ for all n. Hence $n \leq t$ for all n.

Since $\{n\}$ converges to t, there exists M such that

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$n \geq M$ implies $|n - t| < 1$. Thus $t - 1 < n < t + 1$ for $n \geq M$. This means $t < n + 1$ for $n \geq M$ and contradicts the note above. Therefore, F is Archimedean ordered.

Theorem 2.19: If F is an ordered field which satisfies the Bolzano-Weierstrass Sequence Property, then F is Archimedean ordered.

Proof: Assume F is not Archimedean ordered, then as before, the sequence $\{n\}$ is bounded. Hence there exists $\{n_k\}$ a convergent subsequence of $\{n\}$. Let $\lim_{k \to \infty} n_k = t$. As $\{n_k\}$ is monotone increasing, $n_k \leq t$ for all k. Hence, $n \leq t$ for all $n \in N$. For if there exists $M \in N$ such that $M > t$, then $n \geq M$ implies $n > t$. In particular, since $n_M \geq M$, $n_M > t$.

There exists K such that $k \geq K$ implies $|n_k - t| < 1$. That is, $t - 1 < n_k < t + 1$ for $k \geq K$. Thus $t < n_k + 1$ for $k \geq K$. But $n_k + 1 \in N$ and we have a contradiction. Therefore, F is Archimedean ordered.

Theorem 2.20: If F is an ordered field which satisfies the Bolzano-Weierstrass Set Property, then F is Archimedean ordered.

Proof: Assume F is not Archimedean. Then N is bounded in F. Since N is infinite, N has a limit point $t \in F$. Let $I = ]t - 1/2, t + 1/2[$. If $x \in I$, then $t - 1/2 < x < t + 1/2$. Hence $x - 1 < t - 1/2$ and $t + 1/2 < x + 1$. Thus $I \cap N$ may consist of at most one

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element. If $t \in N$, then $I \cap N = \{t\}$ and we have a contradiction. If $t \notin N$, then either $I \cap N = \emptyset$ which is contrary to t being a limit point, or $I \cap N = \{n\}$ where $n \neq t$. In the latter case, let $\delta = |t - n|/2$, then $\delta > 0$. Let $J = ]t - \delta, t + \delta[$. $J \subset I$ with $n \notin J$. Hence $J \cap N = \emptyset$, and we again have a contradiction. This exhausts the possibilities. Therefore, F is Archimedean.

<u>Theorem 2.21</u>: If F is a connected ordered field, then F is Archimedean ordered.

Proof: Assume F is not Archimedean ordered. Let $S = \{x \mid x \in F$ and x is an upper bound for $N\}$ and let $T = F - S$. $S \neq \emptyset$ as F is not Archimedean. $T \neq \emptyset$ as $N \subseteq T$. $S \cup T = F$ and $S \cap T = \emptyset$.

Let $x \in S$ and consider the open interval $I = ]x - 1, x + 1[$. Let $y \in I$. If $y \notin S$, then there exists $n \in N$ such that $y < n$. $x - 1 < y$ implies $x - 1 < n$ and $x < n + 1$. But $n + 1 \in N$, so this contradicts the fact that $x \in S$. Thus $I \subseteq S$ and S is an open set.

Let $z \in T$ and consider $J = ]z - 1, z + 1[$. Let $w \in J$. $z \in T$ implies that there exists $n \in N$ such that $z < n$. Thus $z + 1 < n + 1$ and since $w < z + 1$, $w < n + 1$. $n + 1 \in N$; so $w \in T$ and $J \subseteq T$. Thus T is an open set.

Hence, F is disconnected.

<u>Theorem 2.22</u>: If F is an ordered field which satisfies the Heine-Borel Covering Property, then F is

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Archimedean ordered.

Proof: Let $z \in F$ with $z > 0$. For each $x \in [0, z]$, let $I_x = ]x - 1/2, x + 1/2[$. Then

$$[0, z] \subseteq \cup \{I_x \mid x \in [0, z]\}.$$

Hence there exist $x_1, x_2, \ldots, x_n \in [0, z]$ such that $[0, z] \subseteq \bigcup_{j=1}^{n} I_{x_j}$. We may assume that $x_1 < x_2 < \ldots < x_n$. Then $0 \in I_{x_1}$ and $x_{k+1} - 1/2 \leq x_k + 1/2 < x_{k+1} + 1/2$ for $k = 1, 2, \ldots, n - 1$. For if $x_k + 1/2 < x_{k+1} - 1/2$, then we have $0 \leq x_k + 1/2 < x_{k+1}$, and hence $x_k + 1/2 \in [0,z]$. We also have $x_j + 1/2 \leq x_k + 1/2$ for $j \leq k$, and $x_k + 1/2 < x_j - 1/2$ for $j \geq k + 1$. Hence $x_k + 1/2 \notin I_{x_j}$ for $j = 1, 2, \ldots, n$ which contradicts the fact that $[0, z] \subseteq \bigcup_{j=1}^{n} I_{x_j}$. Therefore, $|x_{k+1} - x_k| \leq 1$ for $k = 1, 2, \ldots, n - 1$.

Now $z \in I_{x_k}$ for some $k$. Thus $z < x_k + 1/2$. However,

$$x_k + 1/2 \leq |x_k + 1/2 - x_k| + |x_k - x_{k-1}| + \ldots + |x_2 - x_1| + |x_1|.$$

Hence, $z < 1/2 + 1 + 1 + \ldots + 1 + 1/2 = k$.

Therefore, $F$ is Archimedean.

We now show that the six properties, Connectedness, the Bounded Monotone Sequence Property, Least Upper Bound Completeness, the Heine-Borel Covering Property, the Bolzano-Weierstrass Set Property, and the Bolzano-Weierstrass Sequence Property, are equivalent. Moreover,

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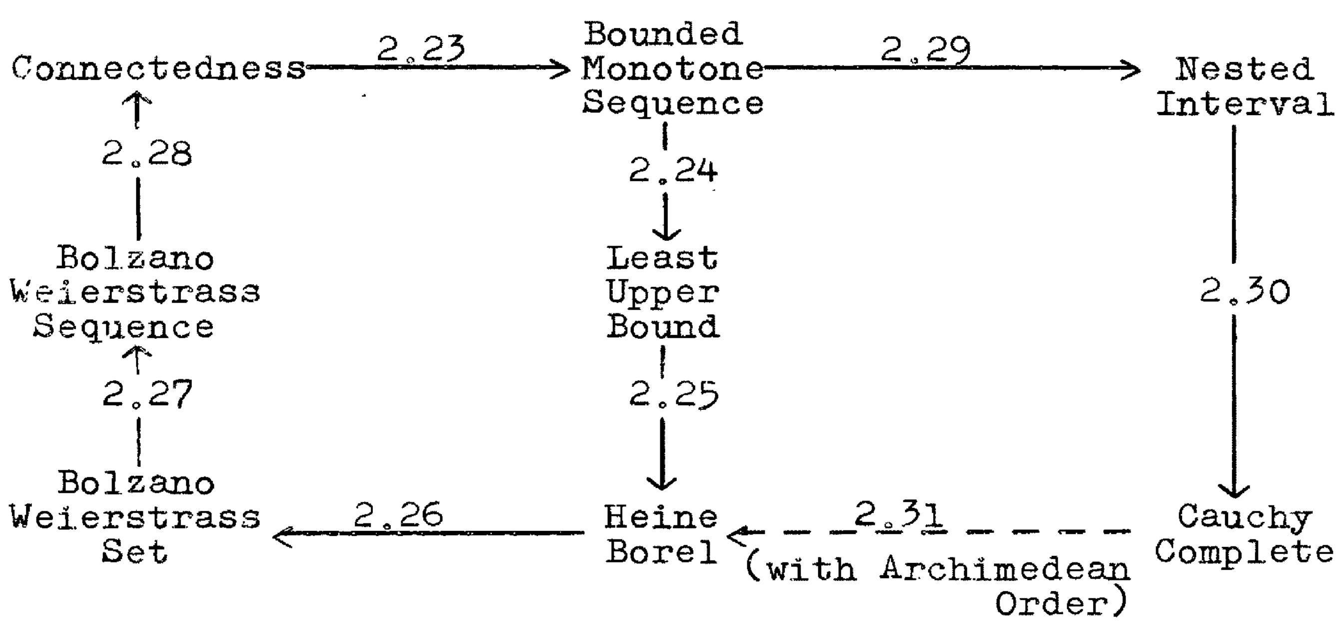
each of these properties implies the Nested Interval Property and Cauchy Completeness. If it is assumed that the field is Archimedean ordered, then the latter two properties imply the first six properties. These assertions will be shown as indicated in the following diagram of implications. A different diagram, given without proof, indicating another method for arriving at these same conclusions, may be found in Buck ([2], p. 391). The numbers refer to the theorem where the indicated implication is shown.




<u>Theorem 2.23</u>: If F is a connected ordered field, then F satisfies the Bounded Monotone Sequence Property.

Proof: Assume F does not satisfy the Bounded Monotone Sequence Property. Let $\{x_n\}$ be a bounded monotone sequence which does not converge. Assume $\{x_n\}$

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is increasing.

Let $A = \{t \mid t \in F$ and $t$ is an upper bound for $\{x_n\}\}$, and let $B = F - A$. Then $A \cup B = F$ and $A \cap B = \emptyset$. $A \neq \emptyset$ since $\{x_n\}$ is bounded. $B \neq \emptyset$ since $x_n \in B$ for each $n$; for if for some $k$, $x_n \leq x_k$ for all $n$, then $\{x_n\}$ converges to $x_k$.

Let $y \in B$. Then there exists $k \in N$ such that $y < x_k$. Let $\delta = x_k - y$. Then $\delta \in F$ and $\delta > 0$. Let $I = ]y - \delta, y + \delta[$. If $x \in I$, then $x < y + \delta = x_k$ and hence $x \in B$. Thus $B$ is open.

Let $w \in A$. Suppose $w \leq t$ for all $t \in A$. Then given $\mu \in F$ with $\mu > 0$, $w - \mu$ is not an upper bound for $\{x_n\}$. Thus there exists $M \in N$ such that $w - \mu < x_M$. Since $\{x_n\}$ is monotone increasing, $n \geq M$ implies $w - \mu < x_n \leq w$. Thus $n \geq M$ implies $|x_n - w| < \mu$. That is, $\{x_n\}$ converges to $w$. This is contrary to our assumption about $\{x_n\}$. Hence, there exists $z \in A$ such that $z < w$. Let $\beta = w - z$ and let $J = ]w - \beta, w + \beta[$. Then $x \in J$ implies $w - \beta < x$; that is, $z < x$. Since $z$ is an upper bound for $\{x_n\}$, $x$ is an upper bound for $x_n$, and hence $x \in A$. Thus $A$ is open.

Therefore, $F$ is disconnected. The proof when $\{x_n\}$ is decreasing is similar in construction.

<u>Theorem 2.24</u>: If $F$ is an ordered field which satisfies the Bounded Monotone Sequence Property, then

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F is Least Upper Bound Complete.

Proof: By theorem 2.18, F is Archimedean. Suppose S is a non-empty subset of F which is bounded above. Let x be an upper bound for S. If $x \in S$, then x is the least upper bound for S and the proof is finished. Suppose $x \notin S$. Then if $w \in S$, there is an $M \in N$ such that $M > x - w$ and hence $w > x - M$. Thus there is an integer $k \geq 0$ such that $x - (k + 1)$ is not an upper bound for S while $x - k$ is an upper bound for S. Let $x_1 = x - k$, and let $I_1 = [x_1 - 1, x_1]$.

Let $x_2$ be the midpoint of $I_1$. If $x_2$ is an upper bound for S, let $I_2 = [x_1 - 1, x_2]$. If $x_2$ is not an upper bound for S and S has only finitely many points between $x_2$ and $x_1$, then clearly S has a least upper bound in F. So in the case where $x_2$ is not an upper bound for S, assume there are infinitely many elements of S between $x_2$ and $x_1$ and let $I_2 = [x_2, x_1]$. Either we inductively choose a nested sequence of closed intervals $I_1 \supseteq I_2 \supseteq \ldots \supseteq I_n \supseteq I_{n+1} \supseteq \ldots$ where $I_n = [a_n, b_n]$, such that each $I_n$ contains infinitely many members of S, $b_n$ is an upper bound for S while $a_n$ is not, and $b_n - a_n = 1/2^{n-1}$; or else the process stops and we have found a least upper bound for S.

Assuming the former we have $x_1 - 1 = a_1 \leq a_n \leq a_{n+1} < b_{n+1} \leq b_n \leq b_1 = x_1$. Thus $\{a_n\}$ is a monotone increasing sequence and $\{b_n\}$ is a monotone decreasing sequence. Both

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are bounded, hence they converge in F. However, recall that $b_n - a_n = 1/2^{n-1}$. Thus, $\lim_{n \to \infty} b_n - a_n = 0$, by corollary 2.16. Therefore, $\lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n$, and we let b be their common value. We proceed to show that b is a least upper bound for S.

If there exists $w \varepsilon S$ such that $b < w$, then $w - b > 0$. Thus there exists an M such that $n \geq M$ implies

$$|b_n - b| = b_n - b < w - b.$$

That is, $b_n < w$ which is contrary to $b_n$ being an upper bound for S. Hence, b is an upper bound for S.

Let z be any upper bound for S. If $z < b$, then $b - z > 0$. Thus there exits K such that $n \geq K$ implies

$$|b - a_n| = b - a_n < b - z.$$

That is, $z < a_n$ which contradicts the fact that $a_n$ is not an upper bound for S. Hence, $b \leq z$ and b is the least upper bound for S. Therefore, F is Least Upper Bound Complete.

Theorem 2.25: If F is a Least Upper Bound Complete ordered field, then F satisfies the Heine-Borel Covering Property.

Proof: Let [a, b] be a closed interval in F and let $C = \{I_\alpha \mid \alpha \varepsilon A\}$ be a collection of open intervals such that $[a, b] \subseteq \bigcup_{\alpha \varepsilon A} I_\alpha$. For each $\alpha \varepsilon A$, let $I_\alpha = ]a_\alpha, b_\alpha[$.

There exists $\alpha \varepsilon A$ such that $a \varepsilon I_\alpha$.

Let $S = \{x \mid x \varepsilon [a, b]$ and $[a, x]$ can be covered with a

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finite subcollection of C}. $S \neq \emptyset$ since $a \in S$. b is an upper bound for S so S has a least upper bound t. Since $a \leq t \leq b$, $t \in [a, b]$. Thus there exists $\delta \in A$ such that $t \in I_\delta$. There must exist $y \in S$ such that $a_\delta < y \leq t$, for otherwise $y \leq a_\delta$ for all $y \in S$ and this contradicts the fact that t is the least upper bound for S.

$y \in S$ implies that there exist $I_j \in C$ for $j = 1, 2, \ldots, n$, such that $[a, y] \subseteq \bigcup_{j=1}^{n} I_j$. Since $[y, t] \subseteq I_\delta$ and $[a, t] = [a, y] \cup [y, t]$, we have $[a, t] \subseteq (\bigcup_{j=1}^{n} I_J) \cup I_\delta$. Thus $[a, t]$ is covered by a finite subcollection of C, and $t \in S$.

Either $t = b$ or $t < b$. If $t < b$, then $t \in ]a_\delta, b_\delta[$ implies there exists $z \in [a, b]$ such that $t < z < b_\delta$. Hence $[a, z] \subseteq (\bigcup_{j=1}^{n} I_j) \cup I_\delta$, and $z \in S$. This is contrary to t being an upper bound for S. Thus $t = b$, and $[a, b]$ is covered by a finite subcollection of C. Hence, F satisfies the Heine-Borel Covering Property.

Theorem 2.26: If F is an ordered field which satisfies the Heine-Borel Covering Property, then F satisfies the Bolzano-Weierstrass Set Property.

Proof: Suppose F does not satisfy the Bolzano-Weierstrass Set Property. Then there exists a bounded infinite set $S \subseteq F$ which has no limit point. S is bounded implies that there exists a closed interval $[a, b]$ such that $S \subseteq [a, b]$. Let $x \in [a, b]$. Since x is not a limit

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point for S, there exists $\mu \varepsilon F$ with $\mu > 0$, such that if $I_x = ]x - \mu, x + \mu[$, then $I_x \cap S$ contains at most the element x.

Let $C = \{I_x \mid x \varepsilon [a, b]\}$. Then $[a, b] \subseteq \bigcup_{x\varepsilon[a,b]} I_x$.

By the Heine-Borel Property, there is a finite subcollection D of C such that $[a, b] \subseteq \cup \{I_x \mid I_x \mid D\}$.
Let $D = \{I_1, I_2, \ldots, I_n\}$. Then since $[a, b] \subseteq \bigcup_{j=1}^{n} I_j$ and $S \subseteq [a, b]$, we have $S \subseteq \bigcup_{j=1}^{n} I_j$.

Thus $S = S \cap (\bigcup_{j=1}^{n} I_j) = \bigcup_{j=1}^{n} (S \cap I_j)$.

By construction, for each j, $S \cap I_j$ contains at most one element. Hence S contains at most n elements. This contradicts the fact that S is infinite. Thus, F does satisfy the Bolzano-Weierstrass Set Property.

<u>Theorem 2.27</u>: If F is an ordered field which satisfies the Bolzano-Weierstrass Set Property, then F satisfies the Bolzano-Weierstrass Sequence Property.

Proof: Let $\{x_n\}$ be a bounded sequence in F. If $S = \{x_n \mid n \varepsilon N\}$ is a finite set, then there exists $r \varepsilon F$ such that $x_n = r$ for infinitely many n, and we can extract a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} = r$ for all k. Hence, $\{x_{n_k}\}$ converges to r.

If S is infinite, then S has a limit point y. Let $n_1$ be the least positive integer such that $0 < |x_{n_1} - y| < 1$.

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Having chosen $n_1$, $n_2$, ..., $n_k$, choose $n_{k+1}$ as follows: let $\delta_k$ = minimum $\{|x_{n_i} - y|, 1/(k + 1) | i = 1, 2, \ldots, k\}$, and let $n_{k+1}$ be the least positive integer such that $0 < |x_{n_{k+1}} - y| < \delta_k$. By the choice of $n_1$, $n_2$, ..., $n_k$, ... $\{n_k\}$ is a strictly increasing sequence of positive integers.

Thus we have defined inductively a subsequence $\{x_{n_k}\}$ of $\{x_n\}$, such that $0 < |x_{n_k} - y| < 1/k$, for each k. Since F satisfies the Bolzano-Weierstrass Set Property, F is Archimedean.

Let $\mu \in F$ with $\mu > 0$. There exists $K \in N$ such that $k \geq K$ implies $1/k < \mu$. Consequently, $k \geq K$ implies $0 < |x_{n_k} - y| < \mu$. Hence $\{x_{n_k}\}$ converges to y and the proof is complete.

Theorem 2.28: If F is an ordered field which satisfies the Bolzano-Weierstrass Sequence Property, then F is connected.

Proof: Suppose F is not connected. Then $F = A \cup B$ where A and B are non-empty, disjoint, open sets.

Let $x_1 \in A$, and $x_2 \in B$. We may assume $x_1 < x_2$.

Let $x_3 = (x_1 + x_2)/2$. If $x_3 \in A$, let $x_4 = (x_3 + x_2)/2$; and if $x_3 \in B$, let $x_4 = (x_3 + x_1)/2$. Having chosen $x_1$, $x_2$, ..., $x_n$, for $n \geq 3$, such that if $x_j \in A$ with $1 \leq j \leq n$ and $x_k \in B$ with $1 \leq k \leq n$, then

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$x_j < x_k$; choose $x_{n+1}$ as follows:

Let $y_n$ = maximum $\{x_j \mid x_j \in A$ for $j = 1, 2, \ldots, n\}$, and $z_n$ = minimum $\{x_j \mid x_j \in B$ for $j = 1, 2, \ldots, n\}$.

Let $x_{n+1} = (y_n + z_n)/2$. Then $y_n < x_{n+1} < z_n$. Hence for $1 \leq j \leq n$, $x_j \in A$ implies $x_j < x_{n+1}$; and for $1 \leq k \leq n$, $x_k \in B$ implies $x_{n+1} < x_k$. Moreover, if $x_j \in A$ with $1 \leq j \leq n + 1$ and $x_k \in B$ with $1 \leq k \leq n + 1$, then $x_j < x_k$.

Thus we obtain the sequence $\{x_n\}$, bounded below by $x_1$ and above by $x_2$, with these properties:

(1) If $x_j \in A$ with $1 \leq j \leq n$ and $x_k \in B$ with $1 \leq k \leq n$, then $x_j < x_k$.

(2) If $x_j \in A$ with $1 \leq j \leq n - 1$, then $x_j < x_n$.

(3) If $x_k \in B$ with $1 \leq k \leq n - 1$, then $x_n < x_k$.

By hypothesis, $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$. Let $\lim_{k \to \infty} x_{n_k} = r$.

We may assume $r \in A$. Since $A$ is an open set, there exists $\delta \in F$ with $\delta > 0$, such that if $I = ]r - \delta, r + \delta[$, then $I \subseteq A$. $\{x_{n_k}\}$ converges to $r$ implies that there exists $M \in N$ such that $k \geq M$ implies $x_{n_k} \in I$. In particular, $x_{n_M} \in I$; so $x_{n_M} \in A$.

Consider $J = \{j \mid j > n_M$ and $x_j \in A\}$. $J \neq \emptyset$ since $n_k \geq k > M$ implies $x_{n_k} \in A$. Let $t$ be the least element in $J$. Then $x_t = (y_{t-1} + z_{t-1})/2$ where $y_{t-1}$ = maximum $\{x_j \mid x_j \in A$ for $j = 1, 2, \ldots, t - 1\}$, and

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$z_{t-1} = \text{minimum } \{x_j \mid x_j \in B \text{ for } j = 1, 2, \ldots, t - 1\}$. Recall that $y_{t-1} < x_t < z_{t-1}$ and that for $1 \leq j < n_M$, $x_j \in A$ implies $x_j < x_{n_M}$. Thus by definition of t and $y_{t-1}$, we have $y_{t-1} = x_{n_M}$. Hence we have $x_t = (x_{n_M} + z_{t-1})/2$, and $x_{n_M} < x_t < z_{t-1}$.

$z_{t-1} \in B$ and $x_{n_M} \in I \subseteq A$ implies

$$r - \delta < x_{n_M} < r + \delta \leq z_{t-1}.$$

Thus $x_t > ((r - \delta) + (r + \delta))/2 = r$.

Since $x_t \in A$, for $n \geq t$ and $x_n \in A$, we have $x_n \geq x_t$. In particular, for $k \geq t$, $x_{n_k} \geq x_t$. Hence, $\lim_{k \to \infty} x_{n_k} \geq x_t$. But $r \geq x_t > r$ is impossible.

Therefore, F is connected.

Theorem 2.28 completes the circle of implications needed to show the equivalence of the first six properties referred to on page 37. We now proceed to show that the Nested Interval Property and Cauchy Completeness are implied by these and that with the assumption of Archimedean ordering all eight are equivalent.

<u>Theorem 2.29</u>: If F is an ordered field which satisfies the Bounded Monotone Sequence Property, then F satisfies the Nested Interval Property.

Proof: Let $\{I_k\}$ be a nested sequence of closed intervals with $I_k = [a_k, b_k]$ for each k. Then

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$a_1 \leq a_k \leq a_{k+1} \leq b_{k+1} \leq b_k \leq b_1$ for $k = 1, 2, 3, \ldots$ . $\{a_k\}$ is a bounded monotone sequence in F and hence converges; let $\lim_{k \to \infty} a_k = y$.

Suppose there exists an M such that $y \notin I_M$. Since $\{a_k\}$ is a monotone increasing sequence, $a_k \leq y$ for all k. Thus $y \notin I_M$ implies $b_M < y$. However, $I_k \subseteq I_M$ for $k \geq M$; so $a_k \leq b_M$ for $k \geq M$. Thus $\lim_{k \to \infty} a_k \leq b_M$, which gives $y \leq b_M < y$, which is impossible. Therefore, $y \in I_k$ for all k; that is, $y \in \bigcap_{k=1}^{\infty} I_k$.

Thus, F satisfies the Nested Interval Property.

<u>Theorem 2.30</u>: If F is an ordered field which satisfies the Nested Interval Property, then F is Cauchy Complete.

Proof: If $\{x_n\}$ is a sequence in F for which there exists an $M \in N$ such that $n \geq M$ implies $x_n = x_M$, then $\{x_n\}$ will be called an <u>essentially constant</u> sequence.

Case 1: If the only Cauchy sequences in F are essentially constant, then these sequences converge, and F is Cauchy Complete.

Case 2: Suppose there exists a Cauchy sequence $\{x_n\}$ in F which is not essentially constant.

(A) We show F has a sequence of positive elements which converges to zero.

Consider the given sequence $\{x_n\}$. For each n, let $y_n = |x_{n+1} - x_n|$. Then $y_n \geq 0$ for all n. $\{x_n\}$ is not

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essentially constant implies that for every $M \in N$ there exists an $n > M$ such that $x_{n+1} \neq x_n$; that is, such that $y_n \neq 0$. For otherwise, $x_{n+1} = x_n$ for all $n \geq M + 1$. Hence, $x_n = x_{M+1}$ for all $n \geq M + 1$.

Let $n_1$ be the least positive integer such that $y_{n_1} \neq 0$. Having chosen $n_1, n_2, \ldots, n_k$ such that $y_{n_j} \neq 0$ for $j = 1, 2, \ldots, k$, let $n_{k+1}$ be the least positive integer greater than $n_k$ such that $y_{n_{k+1}} \neq 0$. Thus we obtain a sequence $\{y_{n_k}\}$ of positive elements of F.

Let $\mu \in F$ with $\mu > 0$. There exists L such that $n, m \geq L$ implies $|x_n - x_m| < \mu$. In particular, $n \geq L$ implies $|x_{n+1} - x_n| < \mu$. Note that $n_k \geq k$ for all k. Hence, $|y_{n_k} - 0| = |y_{n_k}| = |x_{n_k+1} - x_{n_k}| < \mu$ for $k \geq L$. Therefore, $\lim_{k \to \infty} y_{n_k} = 0$.

(B) Let $\{y_n\}$ be any Cauchy sequence in F. We show that $\{y_n\}$ converges. We use the fact that if $c \leq b$ then $[a, b] \cap [c, d]$ is the closed interval $[\max \{a, c\}, \min \{b, d\}]$.

By (A), F has a sequence $\{\delta_n\}$ such that $\delta_n > 0$ for each n and $\{\delta_n\}$ converges to zero. Let $M_1$ be least positive integer such that $n \geq M_1$ implies $|y_n - y_{M_1}| < \delta_1$. Let $a_1 = y_{M_1} - \delta_1$, $b_1 = y_{M_1} + \delta_1$, and $I_1 = [a_1, b_1]$.

Assume that closed intervals $I_1 \supseteq I_2 \supseteq \ldots \supseteq I_j$, and positive integers $M_1 \leq M_2 \leq \ldots \leq M_j$, have been defined,

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such that for each k, $I_k = [a_k, b_k]$, $b_k - a_k \leq 2\delta_k$, and $n \geq M_k$ implies $y_n \in I_k$. Define $M_{j+1}$ and $I_{j+1}$ as follows: Choose $M_{j+1} \geq M_j$ such that $n \geq M_{j+1}$ implies $|y_n - y_{M_{j+1}}| < \delta_{j+1}$.

Let $I_{j+1} = I_j \cap [y_{M_{j+1}} - \delta_{j+1}, y_{M_{j+1}} + \delta_{j+1}]$.

If $n \geq M_{j+1}$, then $n \geq M_j$ which implies $y_n \in I_j$. Moreover, $n \geq M_{j+1}$ implies

$$y_n \in [y_{M_{j+1}} - \delta_{j+1}, y_{M_{j+1}} + \delta_{j+1}].$$

Thus $n \geq M_{j+1}$ implies $y_n \in I_{j+1}$. Let $I_{j+1} = [a_{j+1}, b_{j+1}]$. Then $b_{j+1} - a_{j+1} \leq 2\delta_{j+1}$. Moreover, $I_{j+1} \subseteq I_j$.

Thus we obtain a nested sequence of closed intervals $I_1 \supseteq I_2 \supseteq \ldots \supseteq I_j \supseteq \ldots$, and an increasing sequence of positive integers $M_1 \leq M_2 \leq \ldots \leq M_j \leq \ldots$, such that for each j, $I_j = [a_j, b_j]$, $b_j - a_j \leq 2\delta_j$, and $n \geq M_j$ implies $y_n \in I_j$.

Since F satisfies the Nested Interval Property, $\bigcap_{j=1}^{\infty} I_j \neq \emptyset$. Let $y \in \bigcap_{j=1}^{\infty} I_j$. We show $\{y_n\}$ converges to y.

Let $\mu \in F$ with $\mu > 0$. Since $\{\delta_n\}$ converges to zero, there exists k such that $\delta_k < \mu/2$. $n \geq M_k$ implies $y_n \in I_k$. $y \in \bigcap_{j=1}^{\infty} I_j$ implies $y \in I_k$. Hence, $n \geq M_k$ implies $|y_n - y| \leq b_k - a_k \leq 2\delta_k < \mu$. Thus $\lim_{n \to \infty} y_n = y$.

Therefore, F is Cauchy Complete.

<u>Theorem 2.31</u>: If F is an Archimedean ordered field which is Cauchy Complete, then F satisfies the Heine-Borel

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Covering Property.

Proof: Let $J = [a, b]$ and let $C = \{I_\alpha \mid \alpha \varepsilon A\}$ be a collection of open intervals such that $J \subseteq \bigcup_{\alpha \varepsilon A} I_\alpha$. Assume that no finite subfamily of C covers J.

Let $J_1 = [a_1, b_1]$ be one of the two subintervals, $[a, (a + b)/2]$ or $[(a + b)/2, b]$, such that $J_1$ can not be covered by a finite subfamily of C; since [a, b] is the union of the two subintervals, this has to be true for at least one of them. Then $J_1 \subseteq J$ and $b_1 - a_1 = (b - a)/2$.

Assume $J_k = [a_k, b_k]$ have been defined for $k = 1, 2, \ldots, n$ with $b_k - a_k = (b - a)/2^k$, and $J_{k+1} \subseteq J_k$ for $k = 1, 2, \ldots, n - 1$ such that for $k = 1, 2, \ldots, n$, $J_k$ can not be covered by a finite subfamily of C. Let $J_{n+1}$ be one of the two subintervals, $[a_n, (a_n + b_n)/2]$ or $[(a_n + b_n)/2, b_n]$, such that $J_{n+1}$ can not be covered by a finite subfamily of C. Then $J_{n+1} \subseteq J_n$ and $b_{n+1} - a_{n+1} = (b - a)/2^{n+1}$.

Thus we obtain inductively a nested sequence of closed intervals $\{J_n\}$, such that for each n, $J_n = [a_n, b_n]$ where $b_n - a_n = (b - a)/2^n$, and $J_n$ can not be covered by a finite subfamily of C.

For a fixed k; $n, m \geq k$ implies $b_n, b_m \varepsilon J_k$; hence $|b_n - b_m| \leq (b - a)/2^k$.

F is Archimedean ordered, so by corollary 2.16, we have $\lim_{k \to \infty} 1/2^k = 0$. Hence, $\lim_{k \to \infty} (b - a)/2^k = 0$.

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Let $\mu \in F$ with $\mu > 0$. Then there exists K such that $(b - a)/2^K < \mu$. Thus $n, m \geq K$ implies $|b_n - b_m| < \mu$. Therefore $\{b_n\}$ is a Cauchy sequence and hence converges. Let $y = \lim_{n \to \infty} b_n$. Since $a \leq b_n \leq b$ for all n, $y \in [a, b]$. Consequently there exists $I_\alpha = ]c_\alpha, d_\alpha[ \in C$ such that $y \in I_\alpha$. Let $\delta = \text{minimum } \{|c_\alpha - y|, |d_\alpha - y|\}$. Then $\delta > 0$ and hence there exists $L \in N$ such that $n \geq L$ implies $|b_n - y| < \delta/2$.

We know that $\lim_{n \to \infty} (b_n - a_n) = \lim_{n \to \infty} (b - a)/2^n = 0$; so there exists $M \in N$ such that $n, m \geq M$ implies $|b_n - a_n| = b_n - a_n < \delta/2$.

Let S be the maximum of L and M. $n \geq S$ implies $|y - a_n| \leq |y - b_n| + |b_n - a_n| < \delta$, and $|y - b_n| < \delta/2$. Thus $n \geq S$ implies $[a_n, b_n] \subseteq ]c_\alpha, d_\alpha[$. That is $[a_n, b_n]$ is covered by a finite subfamily of C, namely one interval. This is a contradiction. Thus there does exist a finite subfamily of C which covers J. Therefore, F satisfies the Heine-Borel Covering Property.

One should note that the hypothesis of Archimedean order in theorem 2.31 is necessary. Recall that we showed earlier that the Heine-Borel Covering Property implies Archimedean order, and that R<x> is a non-Archimedean ordered field which is Cauchy Complete.

If F is an Archimedean ordered field which satisfies any one of the eight completeness properties specified on

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page 37, then F is called a complete Archimedean ordered field. The final result of this paper is given in the next theorem which characterizes the complete Archimedean ordered fields.

Theorem 2.32: Let F be a complete Archimedean ordered field. Then there exists $\varphi : F \longrightarrow R$ such that $\varphi$ is an order-preserving field isomorphism of F onto R.

Proof: By theorem 1.28, there exists $\varphi : F \longrightarrow R$ such that $\varphi$ is an order-preserving field isomorphism of F into R. We have only to show that $\varphi$ is onto.

Recall that $\varphi$ is defined by $\varphi(a) = \lim_{n \to \infty} r_n$ where $\{r_n\}$ is a sequence in Q which converges to a in F. If $s \in Q$, then clearly $\varphi(s) = s$.

Let $y \in R$ and let $S = \{x \mid x \in F \text{ and } \varphi(x) < y\}$. R is an Archimedean ordered field so by lemma 1.11 there exists $s \in Q$ such that $s < y$. Since $\varphi(s) = s$, $s \in S$. Hence $S \neq \emptyset$.

There exists $n \in N$ such that $y < n$. $\varphi(n) = n$ so $x \in S$ implies $x < n$ in F. Thus S is a non-empty subset of F which is bounded above. F is a complete Archimedean ordered field implies that S has a least upper bound t. Either $\varphi(t) < y$, $\varphi(t) > y$, or $\varphi(t) = y$.

Assume $\varphi(t) < y$; then there exists $\alpha \in Q$ such that $\varphi(t) < \alpha < y$. $\varphi(\alpha) = \alpha$ implies $\varphi(t) < \varphi(\alpha) < y$. $\varphi(t) < \varphi(\alpha)$ implies $t < \alpha$ in F. $\varphi(\alpha) < y$ implies $\alpha \in S$,

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which implies $\alpha \leq t$. Together these give $\alpha \leq t < \alpha$ which is impossible. Thus $\varphi(t) \geq y$.

Assume $\varphi(t) > y$; then as before there exists $\beta \in Q$ such that $y < \varphi(\beta) < \varphi(t)$. $\varphi(\beta) < \varphi(t)$ implies $\beta < t$. $t$ is the least upper bound for $S$ implies there exists $x \in S$ such that $\beta < x < t$. Hence $\varphi(\beta) < \varphi(x)$. However, $x \in S$ implies $\varphi(x) < y$. Thus we have $\varphi(\beta) < y < \varphi(\beta)$ which is a contradiction.

Therefore, $\varphi(t) = y$, and $\varphi$ is onto.

Theorem 2.32 says that up to isomorphism, R is the only complete Archimedean ordered field. Equivalently, R is essentially the only Archimedean ordered field which is Cauchy Complete.

In conclusion, here is a summary of the main results of this paper. The field Q of rational numbers may be considered a subfield of any ordered field. An Archimedean ordered field is order-isomorphic to a subfield of the field R of real numbers. The six properties: Connectedness, the Bounded Monotone Sequence Property, Least Upper Bound Completeness, the Heine-Borel Covering Property, the Bolzano-Weierstrass Set Property, and the Bolzano-Weierstrass Sequence Property, are equivalent. A complete Archimedean ordered field also satisfies the Nested Interval Property and Cauchy Completeness. These last two properties are equivalent to the first six if

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the field is Archimedean ordered. However, there do exist non-Archimedean ordered fields which are Cauchy Complete. R<x> is such a field. Finally, a complete Archimedean ordered field is order-isomorphic to R.

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